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AN ELEMENTARY PROOF OF THE DESCENT THEOREM FOR GROTHENDIECK TOPOSES

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The key theorem of Joyal & Tierney [1] is the descent theorem for geometric morphisms of Grothendieck toposes (over a fixed base topos $\mathcal{S}$). This theorem says that open surjections are effective descent morphisms – a fact which has remarkable consequences (see *loc. cit.*). Joyal and Tierney prove the descent theorem by first developing descent theory for 'modules' (suplattices) over locales, parallel to descent theory for commutative rings. In this way they provide an algebraic explanation for the theorem. The purpose of this note is to give a direct proof of the descent theorem.

1. Formulation of the descent theorem (see Joyal & Tierney [1])

Let $\mathscr{E} \xrightarrow{f} \mathscr{D}$ be a geometric morphism of Grothendieck toposes over $\mathcal{S}$, and consider the diagram

$$\begin{tikzcd} \mathscr{E} \times_{\mathscr{D}} \mathscr{E} \times_{\mathscr{D}} \mathscr{E} \arrow[r, shift left=2, "p_{12}"] \arrow[r, "p_{23}"] \arrow[r, shift right=2, "p_{13}"'] & \mathscr{E} \times_{\mathscr{D}} \mathscr{E} \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & \mathscr{E} \arrow[r, "f"] & \mathscr{D}. \\ & \mathscr{E} \arrow[u, "\delta"'] & & \end{tikzcd}$$

Descent-data on an object $X \in \mathscr{E}$ consists of a morphism $\theta : p_1^*(X) \to p_2^*(X)$ such that $\delta^*(\theta) = \mathrm{id}$ and $p_{13}^*(\theta) = p_{23}^*(\theta) \circ p_{12}^*(\theta)$ (the cocycle condition). Des(f) denotes the category of pairs (X, θ), θ descent-data on $X \in \mathscr{E}$, where morphisms $(X, \theta) \to (X', \theta')$ are morphisms $X \xrightarrow{f} X'$ in $\mathscr{E}$ which commute with descent-data in the obvious way. Any object $f^*(D)$, $D \in \mathscr{D}$, can be equipped with descent-data in a canonical way, and this gives a commutative diagram

$$\begin{tikzcd} \mathscr{D} \arrow[r] \arrow[dr, "f^*"'] & \mathrm{Des}(f) \arrow[d, "U"] \\ & \mathscr{E} \end{tikzcd}$$

where U is the forgetful functor. f is called an *effective descent morphism* if $\mathscr{D} \to \mathrm{Des}(f)$ is an equivalence of categories. The descent theorem states that every open surjection is an effective descent morphism.

Note that by working inside $\mathscr{D}$, it suffices to prove this theorem for the special case that $\mathscr{E} \xrightarrow{f} \mathscr{D}$ is the canonical geometric morphism $\mathscr{E} \xrightarrow{\gamma} \mathscr{S}$; accordingly, we will only consider this case.

2. Some preliminary remarks

Let $\mathscr{E} = \mathrm{Sh}(\mathbb{C})$, $\mathbb{C}$ a site in $\mathscr{S}$. Then a site for $\mathscr{E} \times \mathscr{E} = \mathscr{E} \times_{\mathscr{S}} \mathscr{E}$ is given by the product-category $\mathbb{C} \times \mathbb{C}$ with the coarsest topology making the projections

$$\mathbb{C} \times \mathbb{C} \underset{P_2}{\overset{P_1}{\rightrightarrows}} \mathbb{C}$$

continuous, i.e. the topology is generated by covers of the form

$$\{(C_i, D) \xrightarrow{(f_i, \mathrm{id})} (C, D)\}_i \quad \text{and} \quad \{(C, D_j) \xrightarrow{(\mathrm{id}, g_j)} (C, D)\}_j,$$

where $\{C_i \xrightarrow{f_i} C\}_i$ and $\{D_j \xrightarrow{g_j} D\}_j$ are covers in $\mathbb{C}$. The inverse image p_1^* of the geometric morphism $\mathscr{E} \times \mathscr{E} \xrightarrow{p_1} \mathscr{E}$ comes from composing with P_1, followed by sheafification. Similarly for p_2^*. The inverse image δ^* of the diagonal $\mathscr{E} \xrightarrow{\delta} \mathscr{E} \times \mathscr{E}$ comes from composing with $\Delta : \mathbb{C} \to \mathbb{C} \times \mathbb{C}$ followed by sheafification: given $Y \in \mathrm{Sh}(\mathbb{C} \times \mathbb{C}) = \mathscr{E} \times \mathscr{E}$, $\delta^*(Y)$ is the sheaf associated to the presheaf $C \mapsto Y(C, C)$. So for $Y = p_1^*(X)$, $\delta^* p_1^*(X) \cong X$, and we have a canonical natural transformation η, $\eta_C : p_1^*(X)(C, C) \to X(C)$, which is the unit of the associated sheaf adjunction. Similarly for p_2^*.

3. The case of connected locally connected geometric morphisms

As a warming up exercise, let us point out that the descent theorem is trivial when $\mathscr{E} \longrightarrow \mathscr{S}$ is connected, locally connected (this is not needed for the proof of the general case). Indeed, let $\mathbb{C}$ be a molecular site for $\mathscr{E}$ (with a terminal, since γ is connected). Constant presheaves on $\mathbb{C}$ are sheaves, and p_1^*, p_2^* are just given by composition with P_1 and P_2 respectively (no sheafification needed). Now suppose X is a sheaf on $\mathbb{C}$, with descent-data $X \circ P_1 \xrightarrow{\theta} X \circ P_2$. This means that we are given functions $\theta_{CD}: X(C) \to X(D)$ for every pair of objects C and D of $\mathbb{C}$. Naturality of θ means that for any $C' \xrightarrow{f} C$, $D' \xrightarrow{g} D$, $X(g) \circ \theta_{CD} = \theta_{C'D'} \circ X(f)$. $\delta^*(\theta) = \mathrm{id}$ means that for any C, $\theta_{CC}: X(C) \to X(C)$ is the identity. And the cocycle condition means that for any triple C, D, E of objects of $\mathbb{C}$, $\theta_{DE} \circ \theta_{CD} = \theta_{CE}$. So in particular, taking $C = E$, θ_{CD} is inverse to θ_{DC}, i.e. θ is an isomorphism. From this it easily follows that X is isomorphic to the constant sheaf $\gamma^*(X(1))$: define

$$X \underset{\varphi}{\overset{\psi}{\rightleftarrows}} \gamma^*(X(1))$$

by the components $\varphi_C = \theta_{1C}$; $\psi_C = \theta_{C1}$. φ and ψ are inverse to each other, and are natural in C by naturality of θ. It remains to show that any morphism $\gamma^*(T) \xrightarrow{\tau} \gamma^*(T')$ which is compatible with the canonical descent-data comes from a map $T \to T'$. But this is clear from the fact that γ^* is full and faithful.

4. A proof of the descent theorem

This is essentially the same as 3, but we have to keep track of sheafification all the time. Let $\mathscr{E} \xrightarrow{\gamma} \mathscr{S}$ be an open surjection, and let $\mathbb{C}$ be an open site for $\mathscr{E}$; i.e. $\mathbb{C}$ has a terminal object 1, and every cover in $\mathbb{C}$ is inhabited. We have to show that

(a) every object $X \in \mathscr{E}$ equipped with descent-data is isomorphic to a constant sheaf;

(b) every morphism $\gamma^*(T) \longrightarrow \gamma^*(T')$ which commutes with the canonical descent-data is of the form $\tau = \gamma^*(f)$.

To prove (a), choose $X \in \mathscr{E}$ with descent-data θ. Write $\mathscr{E} \times \mathscr{E} \xrightarrow{p_i} \mathscr{E}$ and $\mathscr{E} \times \mathscr{E} \times \mathscr{E} \xrightarrow{\bar{p}_i} \mathscr{E}$ for the projections. Identifying $p_2^*(X)(C, D)$ with $p_1^*(X)(D, C)$ in the canonical way, we may regard θ as a system of functions (in $\mathscr{S}$)

$$\theta_{CD}: p_1^*(X)(C, D) \to p_1^*(X)(D, C)$$

which are natural in C, D: for $C' \to C$ and $D' \to D$,

$$\begin{tikzcd}
p_1^*(X)(C, D) \arrow[r, "\theta_{CD}"] \arrow[d] & p_1^*(X)(D, C) \arrow[d] \\
p_1(X)(C', D') \arrow[r, "\theta_{C'D'}"] & p_1^*(X)(D', C')
\end{tikzcd}$$

commutes. This implies that θ_{CD} is determined by its restriction $\theta_{CD} \circ i_1$,

$$X(C) \xhookrightarrow{i_1} p_1^*(X)(C, D) \xrightarrow{\theta_{CD}} p_1^*(X)(D, C)$$

for which we also write θ_{CD}. The condition $\delta^*(\theta) = \mathrm{id}$ means that

$$\begin{tikzcd}
p_1^*(X)(C,C) \arrow[rr, "\theta_{CC}"] \arrow[dr, "\eta_C"'] & & p_1^*(X)(C,C) \arrow[dl, "\eta_C"] \\
& X(C) &
\end{tikzcd}$$

commutes for every C, while the cocycle condition means that

$$\begin{tikzcd}
\bar{p}_1^*(X)(C,D,E) \arrow[r, "\theta_{CD(E)}"] \arrow[d, "\theta_{CE(D)}"] & \bar{p}_1^*(X)(D,C,E) \arrow[d, "\theta_{DE(C)}"] \\
\bar{p}_1^*(X)(E,C,D) \arrow[r, "\sim"] & \bar{p}_1^*(X)(E,D,C)
\end{tikzcd}$$

where $\theta_{CD(E)}$ is the obvious map induced by θ_{CD}, etc.

We will use the following lemma, to be proved below.

Lemma. *For $X \in \mathscr{E} = \mathrm{Sh}(\mathbb{C})$, and objects C, D, E of $\mathbb{C}$, the canonical square*

$$\begin{tikzcd}
p_1^*(X)(C,D) \arrow[r, hook] & \bar{p}_1^*(X)(C,D,E) \\
X(C) \arrow[r, hook] \arrow[u, hook] & p_1^*(X)(C,E) \arrow[u, hook]
\end{tikzcd}$$

is a pullback in $\mathscr{S}$.

Let $S = \{x \in X(1) \mid \theta_{11}(i_1(x)) = i_1(x)\}$, where $i_1 : X(1) \hookrightarrow p_1^*(X)(1,1)$ as above. We claim that $X \cong \gamma^*(S)$ via

$$X \underset{\psi}{\overset{\varphi}{\leftrightarrows}} \gamma^*(S),$$

where φ is the transpose of $S \to X(1)$, and ψ is the map defined by the components

$$\begin{tikzcd}
X(C) \arrow[r, dashed, "\psi_C"] \arrow[d, hook, "i_1"] & \gamma^*(S)(C) \arrow[d, hook, "j_C"] \\
p_1^*(X)(X,1) \arrow[r, "\theta_{C1}"] & p_1^*(X)(1,C)
\end{tikzcd}$$

where j_C is the obvious embedding, natural in C. The nontrivial thing is to show that ψ_C is well-defined, i.e. that $\theta_{c1} \circ i_1$ factors through j_C. (Naturality of ψ_C is then obvious.) So take $x \in X(C)$, and write $y = \theta_{C1}(i_1(x)) \in p_1^*(X)(1,C)$. We have to show that y "locally does not depend on the C-coordinate". y is given as a compatible family $\{y_\alpha\}_\alpha$, $y_\alpha \in X(D_\alpha)$, for a cover $\{(D_\alpha, C_\alpha) \xrightarrow{(D_\alpha, f_\alpha)} (1, C)\}_{\alpha \in \mathscr{A}}$ in $\mathbb{C} \times \mathbb{C}$.

Fix α, and let $x_\alpha = x \upharpoonright f_\alpha \in X(C_\alpha)$. Then $\theta_{C_\alpha D_\alpha}(x_\alpha) = y_\alpha$, and by the cocycle condition, we have for any object E of $\mathbb{C}$ that $\theta_{C_\alpha E}(x_\alpha) = \theta_{D_\alpha E}(y_\alpha)$ in $\bar{p}_1^*(X)(E, D_\alpha, C_\alpha)$. So by the lemma,

$$\theta_{C_\alpha E}(x_\alpha) = \theta_{D_\alpha E}(y_\alpha) \in X(E).$$

Choosing $E = C_\alpha$, we find that $\theta_{C_\alpha C_\alpha}(x_\alpha) \in X(C_\alpha)$, and hence since η_{C_α} is the identity on $X(C_\alpha) \overset{i_1}{\hookrightarrow} p_1^*(X)(C_\alpha, C_\alpha)$, that $\theta_{C_\alpha C_\alpha}(x_\alpha) = x_\alpha$. Now let E run over all the objects D_β, $\beta \in \mathscr{A}$. Clearly by naturality of θ, if

$$\begin{tikzcd} & D_\beta \arrow[dr] & \\ F \arrow[ur, "h"] \arrow[dr, "k"'] & & 1 \\ & D_\gamma \arrow[ur] & \end{tikzcd}$$

then $\theta_{C_\alpha D_\beta}(x_\alpha) \upharpoonright h = \theta_{C_\alpha E}(x_\alpha) = \theta_{C_\alpha D_\gamma}(x_\alpha) \upharpoonright k$, so since $\{D_\beta \to 1\}_{\beta \in \mathscr{A}}$ is a cover in $\mathbb{C}$ (by openness), there is a unique $z_\alpha \in X(1)$ with $z_\alpha \upharpoonright D_\beta = \theta_{C_\alpha D_\beta}(x_\alpha)$. So by naturality of θ again,

$$z_\alpha = \theta_{C_\alpha 1}(x_\alpha) \in X(1),$$

while moreover since $\theta_{C_\alpha C_\alpha}(x_\alpha) = x_\alpha$,

$$z_\alpha \upharpoonright C_\alpha = x_\alpha \in X(C_\alpha).$$

We claim that $\{z_\alpha\}_\alpha$ determines an element $z \in \gamma^*(S)(C)$. (Note that clearly if this is so, $j_C(z) = \theta_{C1}(x)$.) Indeed, the z_α are compatible in the sense that if

$$\begin{tikzcd} & C_\alpha \arrow[dr, "f_\alpha"] & \\ E \arrow[ur, "h"] \arrow[dr, "k"'] & & C \\ & C_{\alpha'} \arrow[ur, "f_{\alpha'}"'] & \end{tikzcd}$$

commutes, then $z_\alpha = z_{\alpha'} \in X(1)$ – this is obvious from naturality of θ. Moreover, each $z_\alpha \in S$. For if E is any object of $\mathbb{C}$, we have $\theta_{1E}(z_\alpha) = \theta_{C_\alpha E}(x_\alpha)$ in $\bar{p}_1^*(X)(1, C_\alpha, E)$ by the cocycle condition, so by the lemma, $\theta_{1E}(z_\alpha) \in X(E)$. Since $\eta_1 | X(1)$ is the identity, we find for $E = 1$ that $\theta_{11}(z_\alpha) = z_\alpha$. This proves that ψ_C is well-defined.

It is now clear that φ and ψ are inverse to each other: One way round, it suffices to show that $\psi_1\varphi_1(s) = s$ for $s \in S$. But $\varphi_1(s) = s \in X(1)$, and $\theta_{11}(s) = i_1(s)$ by definition of S, so this is clear. The other way round, take $x \in X(C)$. Then

$\psi_C(x) \in \gamma^*(S)(C)$ is the element z as above with $z \restriction f_\alpha = z_\alpha \in S \subset X(1)$. So by definition, $\varphi_C(z) \in X(C)$ is given by $\varphi_C(z) \restriction f_\alpha = z_\alpha \restriction C_\alpha$. But $z_\alpha \restriction C_\alpha = x_\alpha$ as we have seen. So $\varphi_C(z) = x$, i.e. $\varphi_C \psi_C = \mathrm{id}$. This proves (a).

To prove (b), suppose $\gamma^*(T) \xrightarrow{\tau} \gamma^*(T')$ is compatible with the canonical descent-data θ and θ' on $\gamma^*(T)$, $\gamma^*(T')$. It is trivial to check that $T = \{t \in \gamma^*(T)(1) \mid \theta_{11}(t) = t\}$, and similarly for T'. So if $t \in T \subset \gamma^*(T)(1)$, then $\theta'_{11}\tau_1(t) = \tau_1(\theta_{11}(t)) = \tau_1(t)$, so $\tau_1(t) \in T'$. Therefore τ comes from a map $T \to T'$, proving (b).

It remains to prove the lemma. To this end, suppose $x \in p_1^*(X)(C, D)$ and $y \in p_1^*(X)(C, E)$ are equal in $\bar{p}_1^*(X)(C, D, E)$. Write $x = \{x_\alpha\}_\alpha$, $x_\alpha \in X(C_\alpha)$ a compatible family for a cover $\mathscr{U} = \{(C_\alpha, D_\alpha) \to (C, D)\}_{\alpha \in \mathscr{A}}$ in $\mathbb{C} \times \mathbb{C}$, and $y = \{y_\beta\}_\beta$, $y_\beta \in X(C_\beta)$, a compatible family for a cover $\mathscr{V} = \{(C_\beta, E_\beta) \to (C, E)\}_{\beta \in \mathscr{B}}$ in $\mathbb{C} \times \mathbb{C}$. Equality of x and y in $\bar{p}_1^*(X)(C, D, E)$ means that there is a common refinement $\mathscr{W} = \{(C_i, D_i, E_i) \to (C, D, E)\}_{i \in I}$ of $\{(C_\alpha, D_\alpha, E) \to (C, D, E)\}_\alpha$ and $\{(C_\beta, D, E_\beta) \to (C, D, E)\}_\beta$ in $\mathbb{C} \times \mathbb{C} \times \mathbb{C}$ on which x and y agree. Replacing $\mathscr{U}$ by $\{(C_i, D_i) \to (C, D)\}$ and $\mathscr{V}$ by $\{(C_i, E_i) \to (C, E)\}_i$ we get the following notationally more manageable situation: we are given $x_i \in X(C_i)$, $y_i \in X(C_i)$, such that whenever we have a commutative diagram

$$\begin{tikzcd} & (C_i, D_i) \arrow[dr] & \\ (A, B) \arrow[ur] \arrow[dr] & & (C, D) \\ & (C_j, D_j) \arrow[ur] & \end{tikzcd}$$

then $x_i \restriction A = x_j \restriction A$, and a similar condition for compatibility of $\{y_i\}$ with D replaced by E. Moreover, since x and y agree on the cover $\mathscr{W}$, $x_i = y_i$ for every i. We now have to show that $x = \{x_i\}$ comes from an element of $X(C)$, i.e. that $\{x_i\}$ is compatible for the cover $\{C_i \to C\}$ in $\mathbb{C}$. So suppose

$$\begin{tikzcd} & C_{i_1} \arrow[dr] & \\ A \arrow[ur] & & C \\ & C_{i_2} \arrow[ul] \arrow[ur] & \end{tikzcd}$$

commutes. Take a cover $\{(P_\alpha, Q_\alpha, R_\alpha) \to (A, D_{i_1}, E_{i_2})\}_\alpha$ refining $\mathscr{W}$; i.e. for each α there is a $j_\alpha \in I$ such that

$$\begin{tikzcd}
(P_\alpha, Q_\alpha, R_\alpha) \arrow[r] \arrow[d] & (A, D_{i_1}, E_{i_2}) \arrow[d] \\
(C_{j_\alpha}, D_{j_\alpha}, E_{j_\alpha}) \arrow[r] & (C, D, E)
\end{tikzcd}$$

commutes. By openness, $\{(P_\alpha, Q_\alpha) \to (A, D_{i_1})\}_\alpha$ is a cover in $\mathbb{C} \times \mathbb{C}$, while moreover,

$$\begin{aligned}
x_{i_1} 1 P_\alpha = x_{j_\alpha} 1 P_\alpha & \quad \text{(by compatibility of } \{x_i\} \text{ over } (C, D)) \\
= y_{j_\alpha} 1 P_\alpha & \quad \text{(by } x = y \text{ over } (C, D, E)) \\
= y_{i_2} 1 P_\alpha & \quad \text{(by compatibility of } \{y_i\} \text{ over } (C, E)) \\
= x_{i_2} 1 P_\alpha & \quad \text{(by } x = y \text{ over } (C, D, E)).
\end{aligned}$$

The family $\{P_\alpha \to A\}_\alpha$ covers A, so $x_{i_1} 1 A = x_{i_2} 1 A$. This completes the proof of the lemma.

Reference

[1] A. Joyal and M. Tierney, An extension of the Galois theory of Grothendieck, Preprint (1982), to appear in Memoirs A.M.S.